Persistent Homology and Spectral Analysis

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Persistent Homology

- A visual and computational tool for examining large data sets (Edelsbrunner and Hare, 2010)
- A way of investigating the evolution of topological features under a changing filtration parameter.

Topological Filtration

- A **filtered topological space** X is a topological space such that X = X_m together with a sequence of subspaces

$$\emptyset = X_1 \subseteq X_2 \subseteq X_3 \subseteq \cdots \subseteq X_m = X$$

Persistent homology

- Associated with the topological filtration is a sequence of inclusions maps ι: $P_i \to P_j$ for $i < j$ such that

$$0 \to X_1 \to X_2 \to \cdots \to X_m = X.$$

Persistent homology

- As usual, we have a sequence of induced homomorphisms

$$0 \to H_n(X_1) \to H_n(X_2) \to \cdots \to H_n(X_m) = H_n(X)$$

induced from the sequence of inclusions maps.

Persistent homology

- The sequence is connected by the homomorphisms $f_n^{i,j}\colon H_n(X_i) \rightarrow H_n(X_j)$
- The images of the homomorphisms are the *n*-th persistent homology groups denoted by $H_n^{i,j}$ = Im$f_n^{i,j}$

Birth and Deaths of Classes

- A homology class γ is said to be born at X_i if $\gamma \notin H_n^{i-1,j}$. A homology class γ born at X_i is said to die entering X_j if it merges with an older class while proceeding from X_{j-1} to X_j.
- This process is called the Elder Rule.

Birth and Deaths of Classes

Purple = Image of Maps

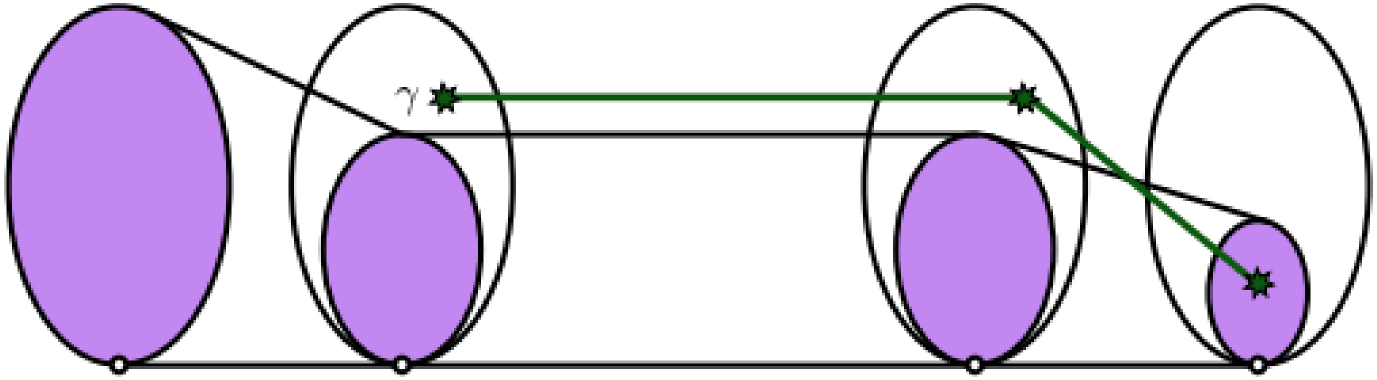


FIGURE 2.1: The class γ is born at $\mathbb{X}_i$ because it is not in the image of the map from $H_p(\mathbb{X}_{i-1})$. It dies entering $\mathbb{X}_j$ because it was still not in the image of $H_p(\mathbb{X}_{i-1})$ at $H_p(\mathbb{X}_{j-1})$, but has merged with an older class upon entering $H_p(\mathbb{X}_j)$.

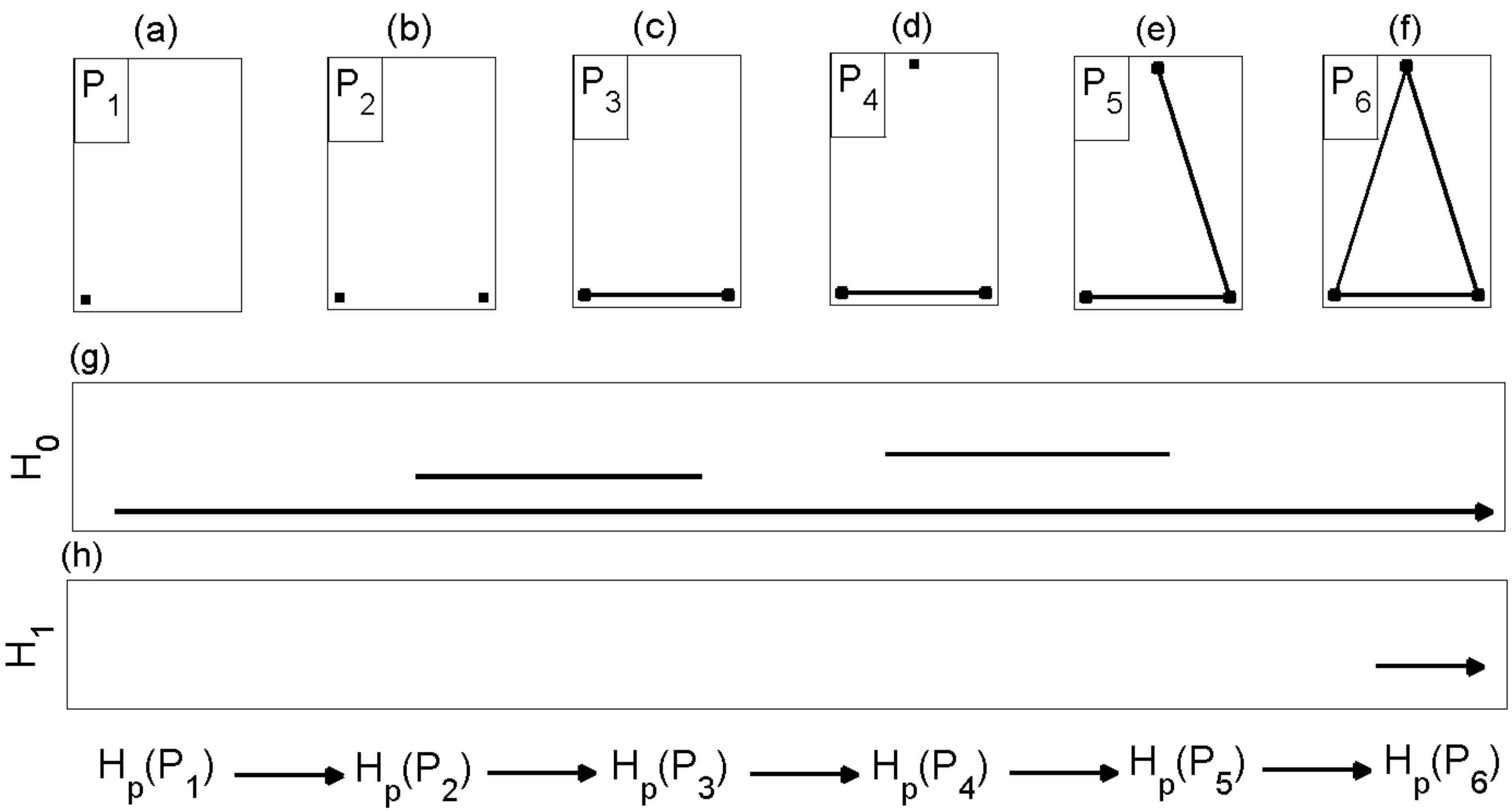
Persistent indices

- For a homology class that is born at X_i and that dies entering X_j, there is a persistence index, which is defined as *j* - *i* and is called the index of persistence of the class (Edelsbrunner and Harer, 2010).

Barcodes

- An arbitrary horizontal collection of lines representing the birth and deaths of homology classes
- A visual tool

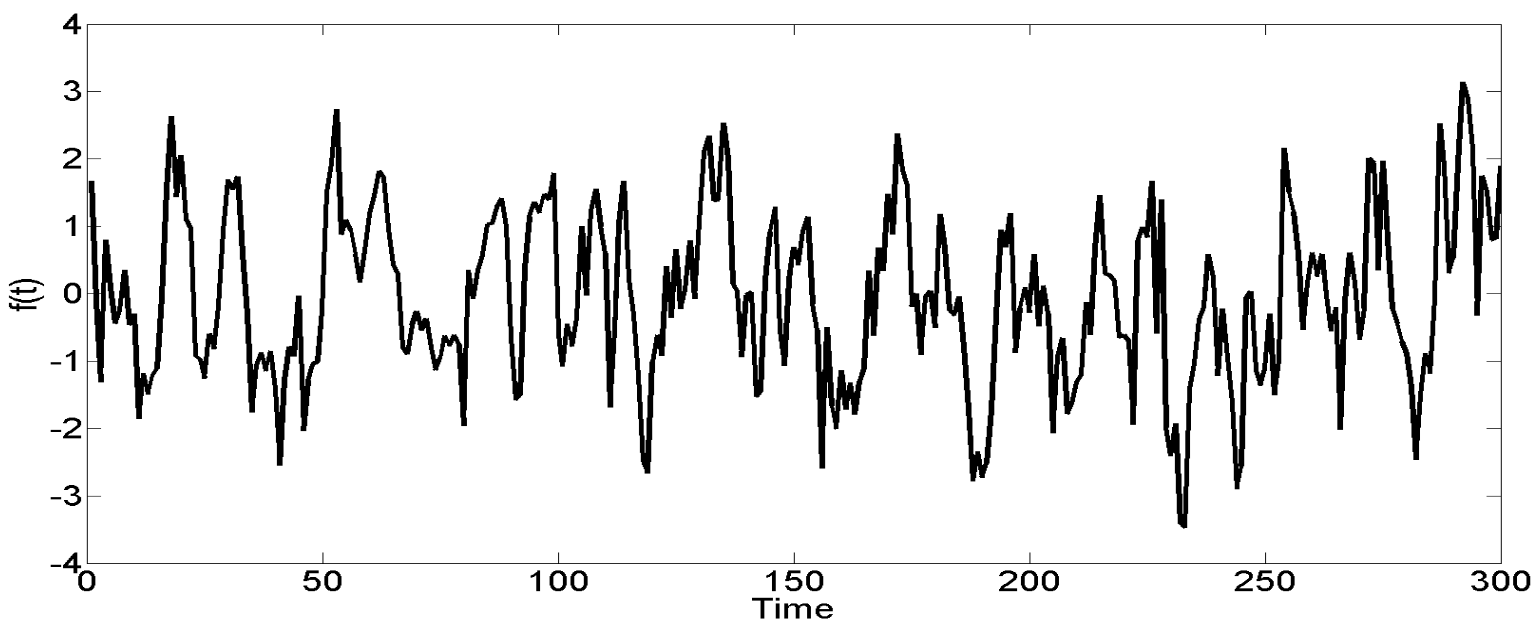
Example



An application of persistent homology

Time series analysis

- Suppose we are given a time series *f*(t). Are there any interesting features in the time series.
- Does the time series contain physically meaningful information or is the time series a random function of time?



Fourier Transform

- One way to extract information from a time series is to use a Fourier transform

$$\widehat{f(\omega)} = \int_{-\infty}^{\infty} f(t)\, e^{-2\pi i t \omega} dt$$

to decompose the variance of the time series as a function of frequency

Wavelet Transform

- Another transform is the wavelet transform, which can be use to decompose the variance of a time series as a function of time and frequency.

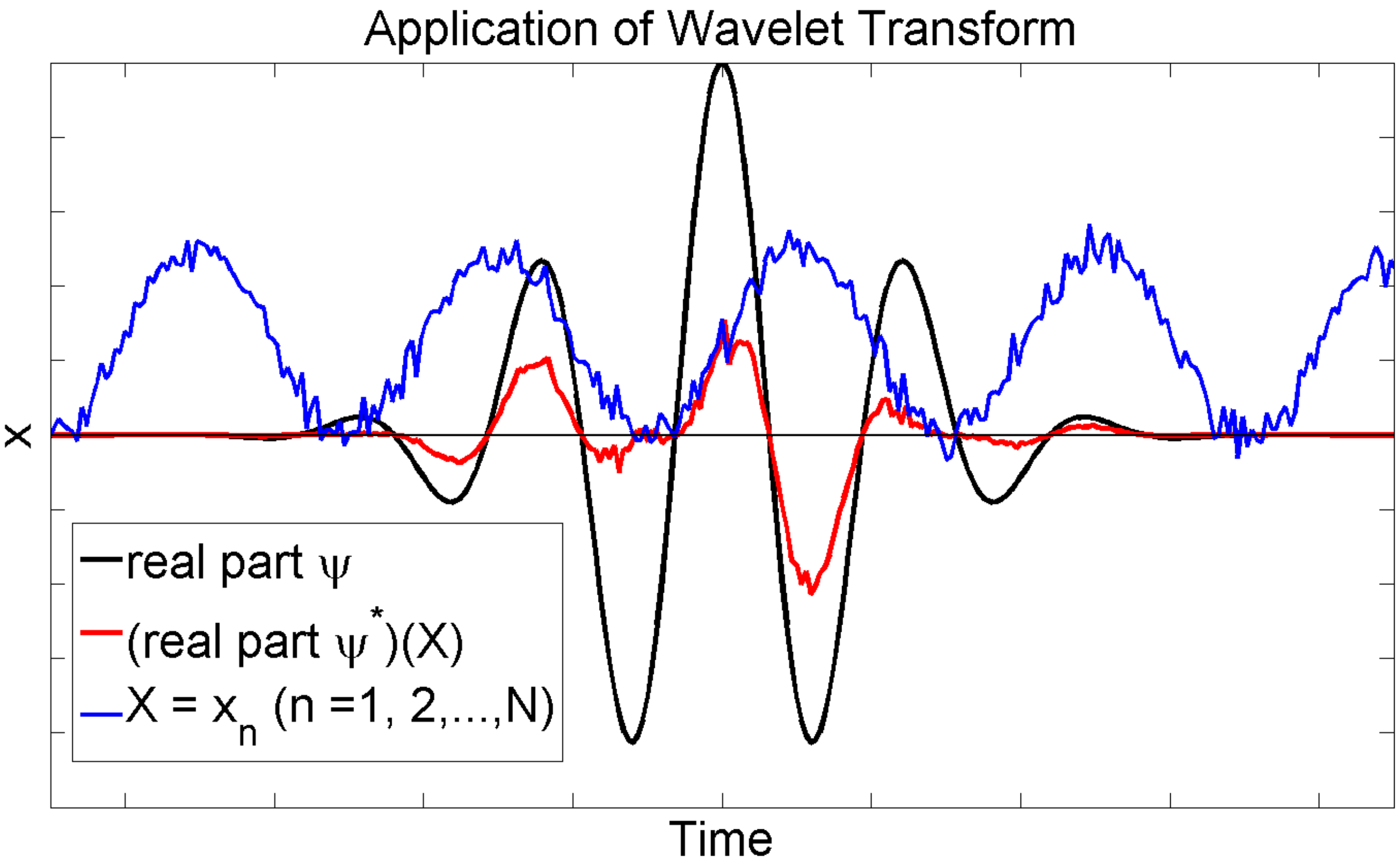
$$X(b, s) = \frac{1}{\sqrt{s}} \int_{-\infty}^{\infty} \psi \overline{\left(\frac{t - b}{s}\right)} f(t)dt$$

- Wavelet power is define as $|X(b, s)|^2$

Morlet Wavelet

- $\psi_0(\eta) = \pi^{-1/4} e^{i\omega_0\eta} e^{-\frac{1}{2}\eta^2}$
- ω_0 is the dimensionless frequency,
- $\eta = s \cdot t$,
- s is the wavelet scale
- t is time

The Wavelet Transform



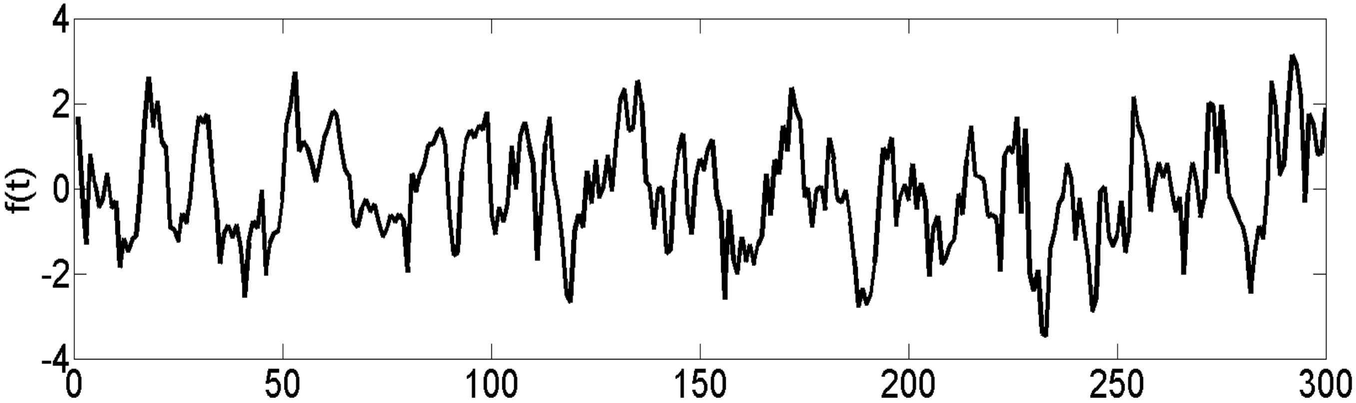
Significance Testing

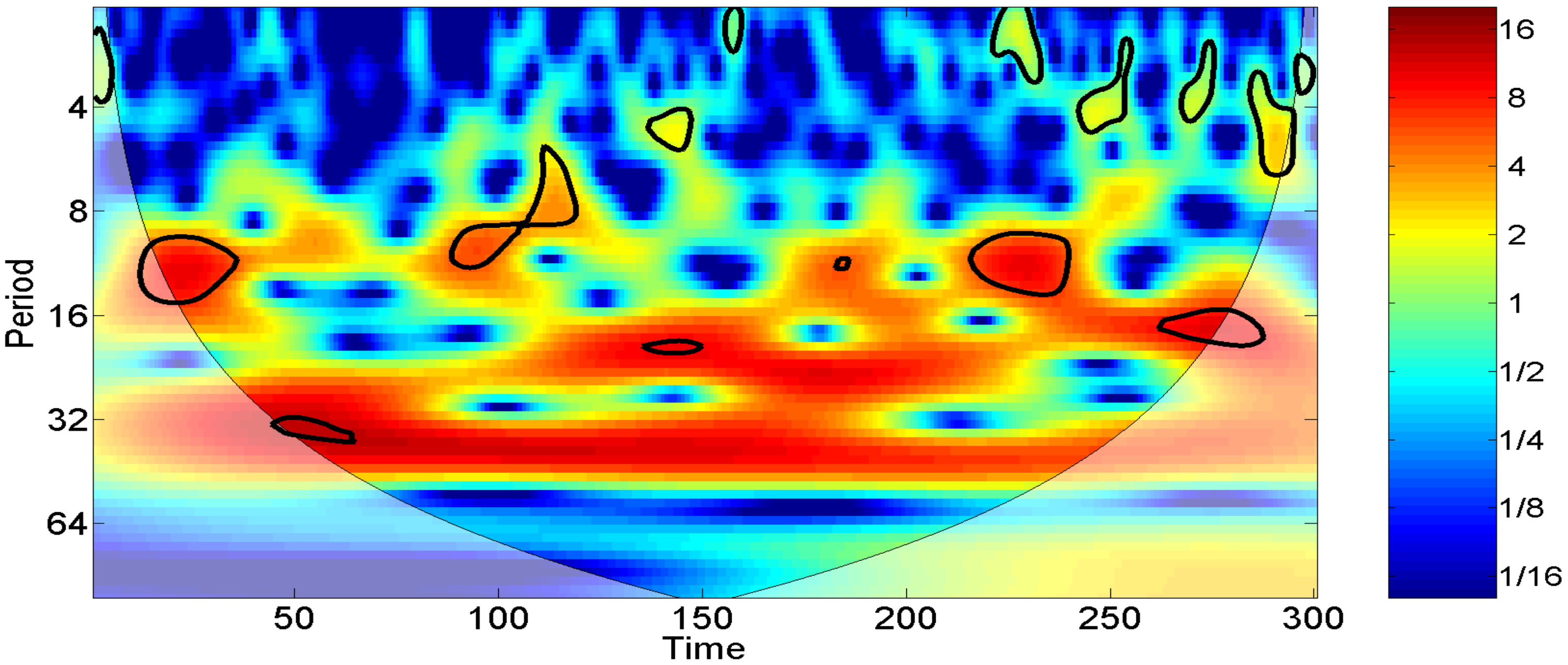
- Random fluctuations can produce large wavelet power.
- How can we distinguish random fluctuations from potentially physically meaningful structures?

Significance Testing

- One way to address the problem is to assign to each wavelet power coefficient a probability ***p*** that the wavelet power did not arise from a random (stochastic) fluctuation when a null hypothesis is true (Torrence and Compo, 1998).

The Wavelet Power Spectrum





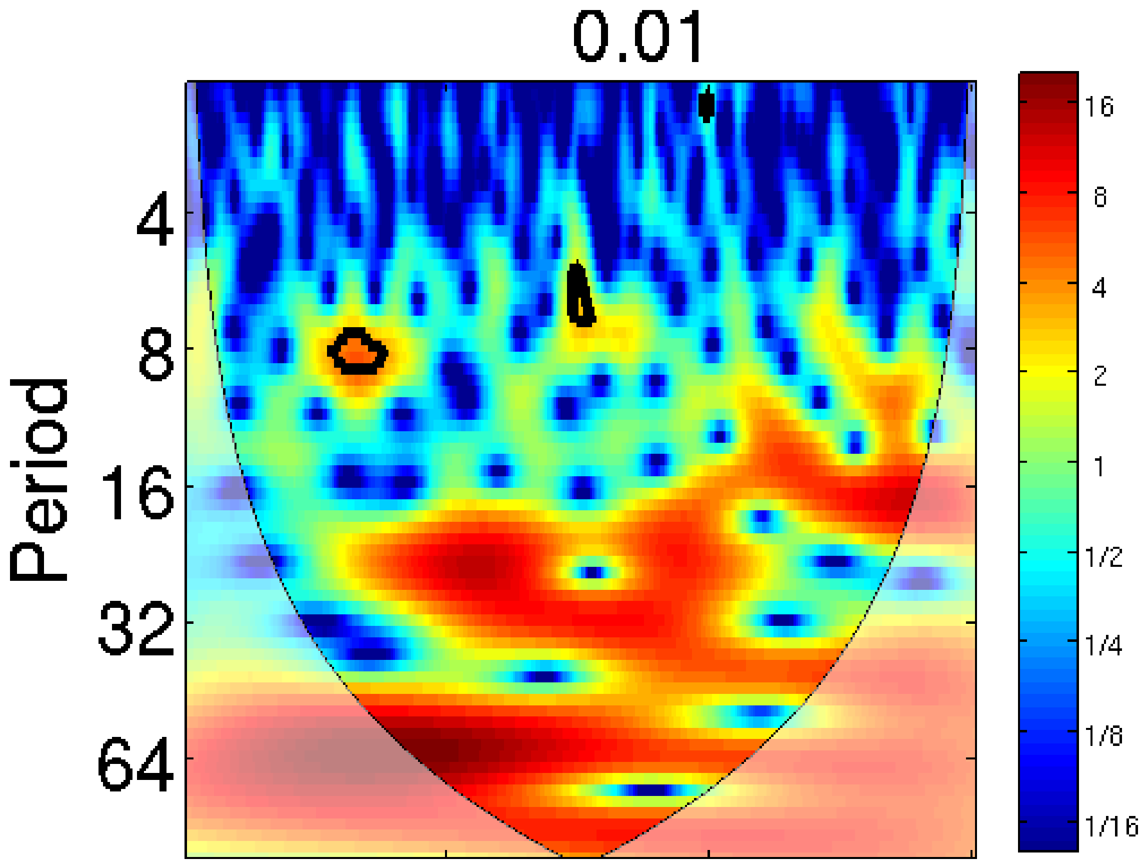
Significance Patches

- The sets $P_i = \{(b, s) \colon p \leq p_i, \}$, where $p_i < p_j$ for $i < j$ are subspaces of $\mathbb{H} \subset \mathbb{R}^2$ and thus we have

$$0 \to H_n(P_1) \to H_n(P_2) \to \cdots \to H_n(P_m) = H_n(\mathbb{H})$$

- We can now calculate persistent homology in the present setting

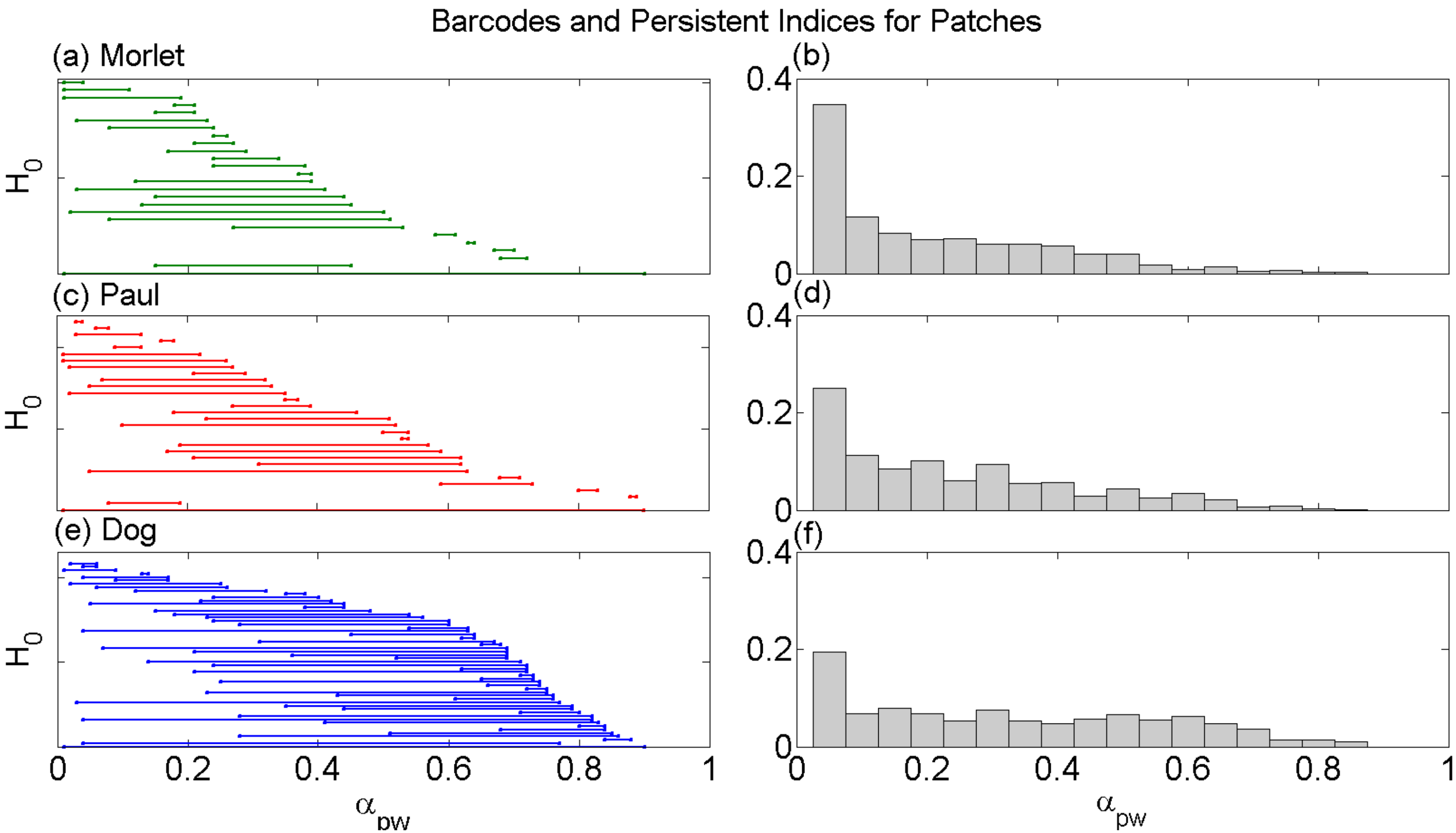
Persistent Homology of Noise



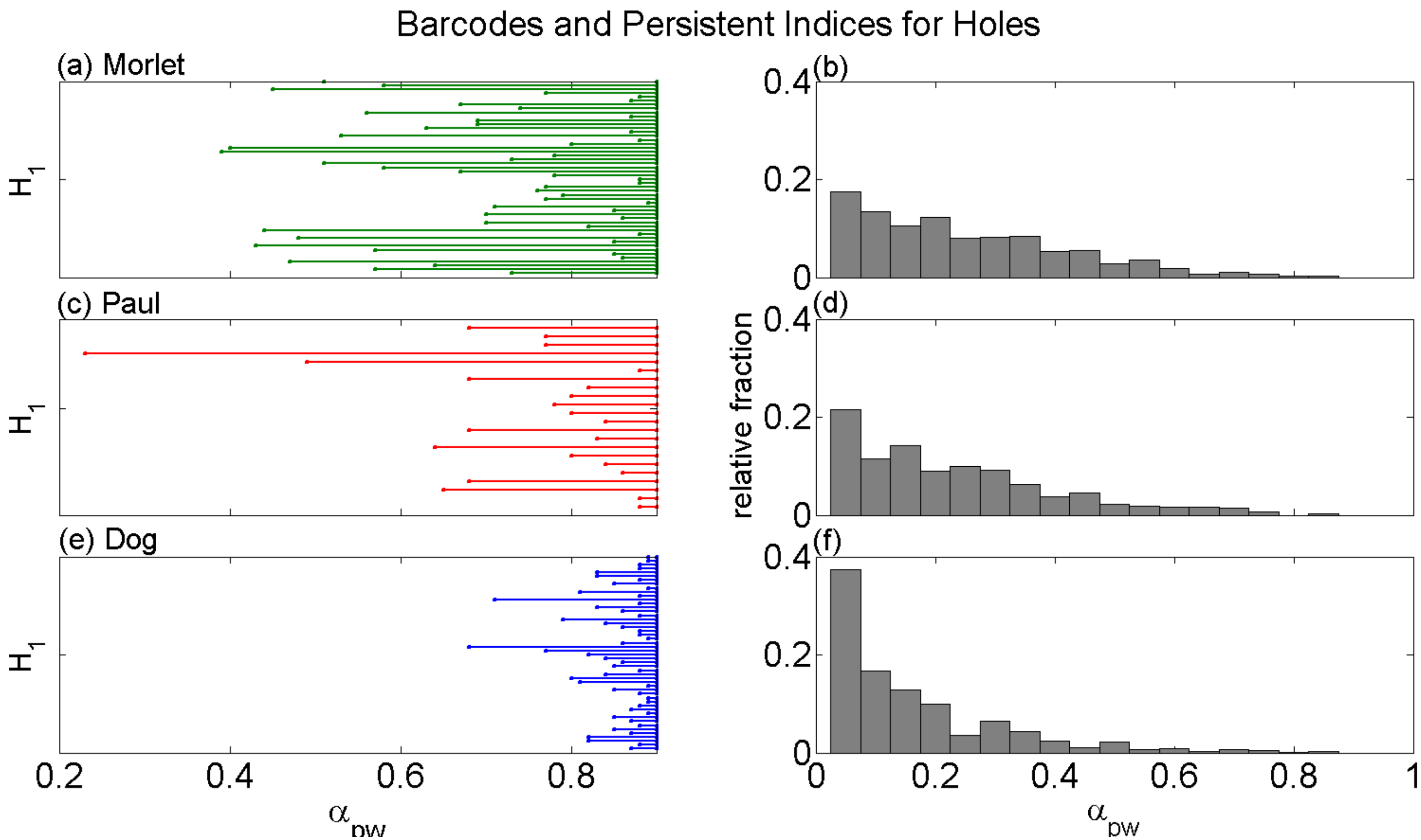
Persistent Homology of noise

- Lets generate a large ensemble of random time series and corresponding wavelet power spectra to see if there are topological patterns

Barcodes for noise



Barcodes for noise



Conculsions

- Persistent homology can help understand properties of random time series.
- Noise has topological signatures in wavelet power spectra.